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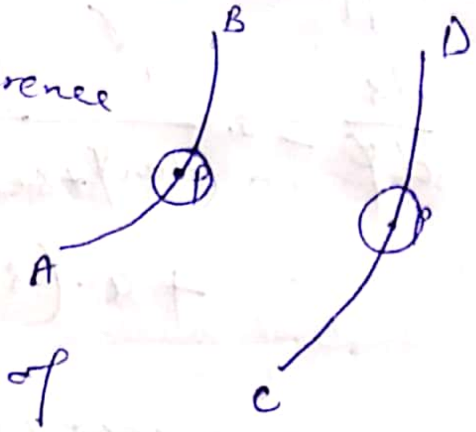
Curvature :-

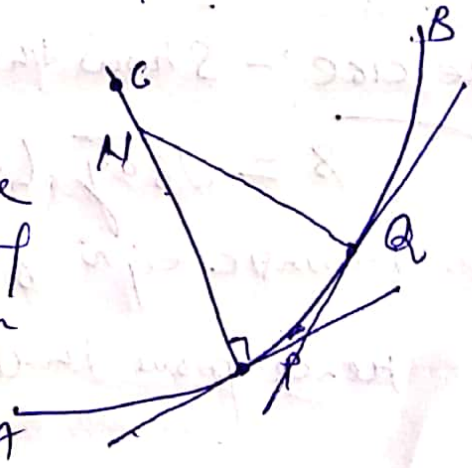
The geometrical difference between AB and CD is expressed by saying that the curvature of AB is greater than that of CD.



If a small arc in the neighbourhood of P on the curve AB and CD is regarded as circular, then it is quite clear that radius of the circle at point P on AB is lesser than that on CD. i.e. greater the curvature lesser the radius and viceversa. This forms the basis of curvature.

Let AB be a curve and P and Q are two points on AB that are neighbouring points on the curve AB. Let the normals drawn at the point P and Q are intersect to each other at the point N; as shown in figure.

If N tends to a definite point C as $Q \to P$, then the point C is called centre of curvature at the point P on the curve AB. The length CP is called radius of curvature of the curve at point P and is denoted by ρ. The reciprocal of the distance CP; i.e. $\frac{1}{CP}$ i.e. $\frac{1}{\rho}$ is called the curvature of the curve at P.

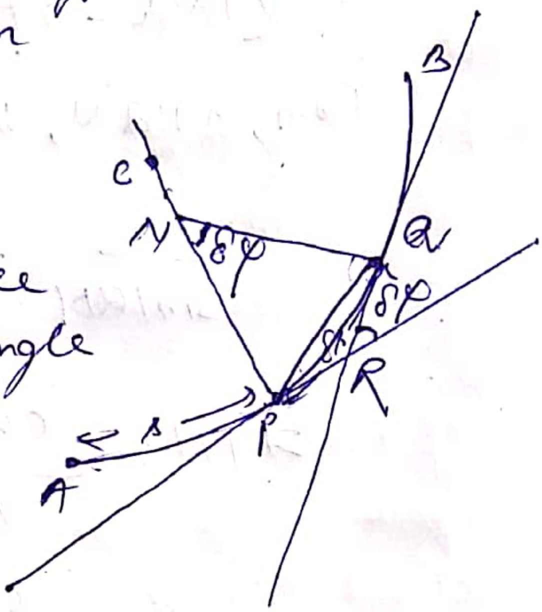


A circle with C as centre and CP as radius is called the circle of curvature of the curve at P.

Alternative definition of curvature:

Let P and Q be two neighbouring points on the curve AB. Let $\delta\varphi$ be the angle between tangents at the point P & Q on the curve AB. Then $\delta\varphi$ is called angle of contingence of the arc PQ. It is also angle between normal at points P and Q on the curve AB.

It is also called the total curvature of arc PQ.



The fraction $\dfrac{\text{angle of contingence}}{\text{length of arc}} = \dfrac{\delta\varphi}{\delta s}$ is called the average curvature of the arc PQ. The limiting value of the average curvature of the curve at the point P when Q tends to P is called the curvature.

Thus, curvature at the point P

$$= \lim_{Q \to P} \frac{\delta\varphi}{\delta s}$$

$$= \lim_{\delta s \to 0} \frac{\delta\varphi}{\delta s} = \frac{d\varphi}{ds}$$

and radius of curvature at the point P,

$$= \frac{1}{\text{curvature}} = \frac{ds}{d\varphi} = \rho$$

Formula for Radius of curvature in Intrinsic form:

A relation between s and φ is called the intrinsic eqn of the curve. The formula $\rho = \dfrac{ds}{d\varphi}$ is used to find ρ, when the eqn of the curve is given in intrinsic form.

By previous diagram, we have.

$\angle PNQ = \delta\varphi$.

From, $\triangle PQN$, by sine formula, we have,

$$\frac{PN}{\sin PQN} = \frac{\text{Chord } PQ}{\sin\delta\varphi}.$$

$$\Rightarrow PN = \frac{\text{chord } PQ}{\sin\delta\varphi} \cdot \sin PQN.$$

$$= \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\varphi} \cdot \frac{\delta\varphi}{\sin\delta\varphi} \cdot \sin PQN.$$

Let ρ be the radius of curvature at P, then

$$\rho = \lim_{Q\to P} PN.$$

$$= \lim_{\delta s\to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\varphi} \cdot \frac{\delta\varphi}{\sin\delta\varphi} \cdot \sin PQN$$

$$= 1 \cdot \frac{ds}{d\varphi} \cdot 1 \cdot \sin\frac{\pi}{2} = \frac{ds}{d\varphi}.$$

$$\rho = \frac{ds}{d\varphi}.$$

$$\text{curvature} = \frac{1}{\rho} = \frac{d\varphi}{ds}.$$

Example :- Find the radius of curvature at the point (s, φ) on the curve.

$$s = a\log(\tan\varphi + \sec\varphi) + a\tan\varphi\sec\varphi.$$

Soln :- We have

$$s = a\log(\tan\varphi + \sec\varphi) + a\tan\varphi\cdot\sec\varphi.$$

Now differentiating w.r.t. φ; we have

$$\frac{ds}{d\varphi} = a \cdot \frac{1}{(\tan\varphi + \sec\varphi)} \cdot (\sec^2\varphi + \sec\varphi \cdot \tan\varphi)$$

$$+ a \cdot (\sec^2\varphi \cdot \sec\varphi + \tan\varphi \cdot \sec\varphi \tan\varphi)$$

$$= a \sec\varphi \frac{(\cancel{\sec\varphi + \tan\varphi})}{(\cancel{\sec\varphi + \tan\varphi})}$$

$$+ a \sec\varphi (\sec^2\varphi + \sec^2\varphi - 1)$$

$$= \cancel{a \sec\varphi} + 2a \sec^3\varphi - \cancel{a \sec\varphi}$$

$$\frac{ds}{d\varphi} = 2a \sec^3\varphi = \rho \text{ radius of curvature}$$

so curvature is $\frac{1}{2a\sec^3\varphi}$

Exercise :- Show that the curve for which

$s = \sqrt{8ay}$ (the cycloid) has for its

Intrinsic eqn $s = 4a \sin\varphi$.

Hence prove that $\rho = 4a\sqrt{1 - \frac{y}{2a}}$.