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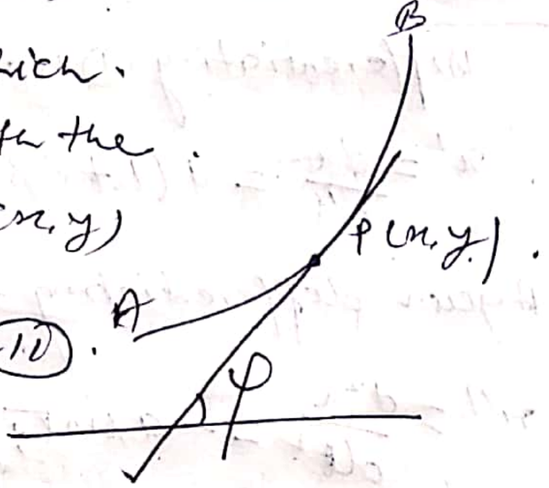
B.Sc. (Paper IInd)
Lecture-2

Cartesian Formula for Radius of curvature :-

Eqn of the curve is given by
$$y = f(x) \quad \text{--- (1)}$$



Let φ be the angle which the tangent makes with the x-axis at any point $P(x, y)$

Then $\tan\varphi = \dfrac{dy}{dx}$ --- (11)

Differentiating eqn (11) w.r.t. s, both sides, we have

$$\sec^2\varphi \cdot \frac{d\varphi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \sec^2\varphi \cdot \frac{1}{\rho} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \cdot \frac{dx}{ds}$$

$$\Rightarrow \sec^2\varphi \cdot \frac{1}{\rho} = \frac{d^2y}{dx^2} \cdot \cos\varphi \qquad \left(\cos\varphi = \frac{dx}{ds}\right)$$

$$\Rightarrow \rho = \frac{\sec^3\varphi}{(d^2y/dx^2)} = \frac{(\sec^2\varphi)^{3/2}}{(d^2y/dx^2)}$$

$$\rho = \frac{(1+\tan^2\varphi)^{3/2}}{(d^2y/dx^2)} = \frac{\{1+(dy/dx)^2\}^{3/2}}{(d^2y/dx^2)}$$

Note:- This formula does not hold good when the tangent is parallel to y-axis. In this case radius of curvature is given by
$$\rho = \left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2} \Big/ \frac{d^2x}{dy^2}$$

Examples:- Findout the radius of curvature at the point (x, y) of the parabola. $y^2 = 4ax$.

Soln:- We have given curve

$$y^2 = 4ax \quad \text{--- (i)}$$

Differentiating w.r.t. x, we have

$$2y \frac{dy}{dx} = 2a \cdot 2$$

$$\Rightarrow \frac{dy}{dx} = (2a/y) \quad \text{--- (ii)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{\sqrt{4ax}} = \sqrt{\frac{4a^2}{4ax}} = \frac{\sqrt{a}}{\sqrt{x}} \quad \text{(iii)}$$

Again differentiating w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \sqrt{a} \cdot \left(-\frac{1}{2}\right) \cdot x^{-3/2} = -\frac{\sqrt{a}}{2x^{3/2}} \quad \text{(iv)}$$

We know that radius of curvature

$$\rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \left(\frac{d^2y}{dx^2}\right)$$

$$= \left[1 + \left(\frac{\sqrt{a}}{\sqrt{x}}\right)^2\right]^{3/2} \Big/ \left(-\frac{\sqrt{a}}{2x^{3/2}}\right)$$

$$= \frac{(a+x)^{3/2} / x^{3/2}}{-\sqrt{a}/2 \cdot x^{3/2}} = -\frac{2(a+x)^{3/2}}{\sqrt{a}}.$$

$$\rho = \frac{2(a+x)^{3/2}}{\sqrt{a}}.$$

Parametric Formula for radius of curvature:

Let the eqn of the curve in parametric form is given by $x = f(t)$; $y = g(t)$. —(i)

Then $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \quad \text{—(ii)}$$

$$\frac{dy}{dx} = \frac{y'}{x'} \quad \text{—(iii)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y'}{x'}\right)$$

$$= \frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx}$$

$$= \left(\frac{x'y'' - x''y'}{x'^2}\right)\Big/ x'$$

$$\frac{d^2y}{dx^2} = (x'y'' - x''y')/x'^3 \quad \text{—(iv)}$$

Using eqn (iii) and (iv) in $\rho = \dfrac{[1+(dy/dx)^2]^{3/2}}{(d^2y/dx^2)}$ we have

$$\rho = \left[1 + \frac{y'^2}{x'^2}\right]^{3/2} \Big/ \left\{(x'y'' - x''y')/x'^3\right\}$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$, where dashes denotes number of derivative of x & y w.r.t. t.

which is required eqn.

Example :- Prove that the radius of curvature at any point of the cycloid $x = a(t + \sin t)$; $y = a(1 - \cos t)$ is given by $\rho = 4a \cos t/2$.

Soln :- We have $x = a(t + \sin t)$; $y = a(1 - \cos t)$ —— (I)

Differentiating (I) w.r.t. t, we have

$$x' = \frac{dx}{dt} = a(1 + \cos t), \; y' = \frac{dy}{dt} = a \sin t \quad \text{—— (II)}.$$

Again differentiating eqn (II) w.r.t. 't', we have

$$x'' = \frac{d^2x}{dt^2} = -a \sin t; \; y'' = \frac{d^2y}{dt^2} = a \cos t \quad \text{—— (III)}$$

Using eqn (II) & (III) in $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{(x'y'' - x''y')}$

$$\rho = \frac{\left[a^2(1+\cos t)^2 + a^2 \sin^2 t\right]^{3/2}}{a(1+\cos t) \cdot a \cos t + a \sin t \cdot a \sin t}$$

$$= \frac{\left[a^2 + a^2 \cos^2 t + 2a^2 \cos t + a^2 \sin^2 t\right]^{3/2}}{a^2 \cos t + a^2 \cos^2 t + a^2 \sin^2 t}$$

$$= \left[2a^2(1+\cos t)\right]^{3/2} \Big/ a^2(1+\cos t)$$

$$= \left[2a^2 \cdot 2 \cdot \cos^2 \frac{t}{2}\right]^{3/2} \Big/ a^2 \, 2 \cdot \cos^2 t/2$$

$$= \left[(2a \cos t/2)^2\right]^{3/2} \Big/ 2a^2 \cos^2 t/2$$

$$= (2a \cos t/2)^3 \Big/ 2(a \cos t/2)^2$$

$\rho = 4a \cos t/2$ which is required.

(Hence proved).