Arvind Kr. Yadav
Raja Singh College Siwan

B. Sc. I
Differential Calculus
(Paper IInd)
Lecture- 3.

Formula for radius of curvature when x & y are given to be function of s :-

Let us consider $x = f(s)$ and $y = \phi(s)$ —— (i)

By tangents and normals; we know that

$$\cos\psi = \frac{dx}{ds} \text{ and } \sin\psi = \frac{dy}{ds} \quad \text{—— (ii)}$$

Differentiating above eqn w.r.to s; we have

$$-\sin\psi \frac{d\psi}{ds} = \frac{d^2x}{ds^2} \;\&\; \cos\psi \cdot \frac{d\psi}{ds} = \frac{d^2y}{ds^2} \quad \text{—— (iii)}$$

squaring both sides and adding, we have

$$(\sin^2\psi + \cos^2\psi)\left(\frac{d\psi}{ds}\right)^2 = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$$

$$\Rightarrow \left(\frac{d\psi}{ds}\right)^2 = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 \quad \text{—— (iv)}$$

$$\Rightarrow \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 \quad \text{—— (v)}$$

Example :- Prove that radius of curvature of the curve $x = a\cos\left(\frac{s}{a}\right)$; $y = a\sin\left(\frac{s}{a}\right)$ is a.

Solution :- Given that $x = a\cos\left(\frac{s}{a}\right)$; $y = a\sin\left(\frac{s}{a}\right)$ —— (i)

Differentiating w.r.t. s; we have

$$\frac{dx}{ds} = a \cdot \frac{1}{a}\left(-\sin\left(\frac{s}{a}\right)\right); \quad \frac{dy}{ds} = a \cdot \frac{1}{a} \cos\left(\frac{s}{a}\right)$$

$$\frac{dx}{ds} = -\sin\left(\frac{s}{a}\right); \quad \frac{dy}{ds} = \cos\left(\frac{s}{a}\right) \quad \text{— (ii)}$$

Again differentiating w.r.t. s, we have

$$\frac{d^2x}{ds^2} = -\frac{1}{a}\cos\left(\frac{s}{a}\right); \quad \frac{d^2y}{ds^2} = -\frac{1}{a}\sin\left(\frac{s}{a}\right) \quad \text{— (iii)}$$

Now squaring and adding eqn (iii), we have

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 = \frac{1}{a^2}\left\{\cos^2\left(\frac{s}{a}\right) + \sin^2\left(\frac{s}{a}\right)\right\}$$

$$\Rightarrow \frac{1}{\rho^2} = \frac{1}{a^2} \Rightarrow \rho = a.$$

(Hence proved).

- Formula for radius of curvature when x and y are given to be function of ψ:

We know that $\cos\psi = \frac{dx}{ds}$ & $\sin\psi = \frac{dy}{ds}$ — (i)

$$\Rightarrow \cos\psi = \frac{dx}{ds} = \frac{dx}{d\psi}\cdot\frac{d\psi}{ds} = \frac{dx}{d\psi}\cdot\frac{1}{\rho} \quad \text{— (ii)}$$

$$\& \sin\psi = \frac{dy}{ds} = \frac{dy}{d\psi}\cdot\frac{d\psi}{ds} = \frac{dy}{d\psi}\cdot\frac{1}{\rho} \quad \text{— (iii)}$$

squaring eqn (ii) & (iii) and adding, we have

$$\rho^2 = \left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2$$

Example: Prove that for any curve.

$$\frac{1}{\rho} = \frac{d}{dx}\left(\frac{dy}{ds}\right).$$

Solution: We are given that $\frac{d}{dx}\left(\frac{dy}{ds}\right)$

$$= \frac{d}{dx}(\sin\psi) = \frac{d}{d\psi}(\sin\psi) \cdot \frac{d\psi}{dx}.$$

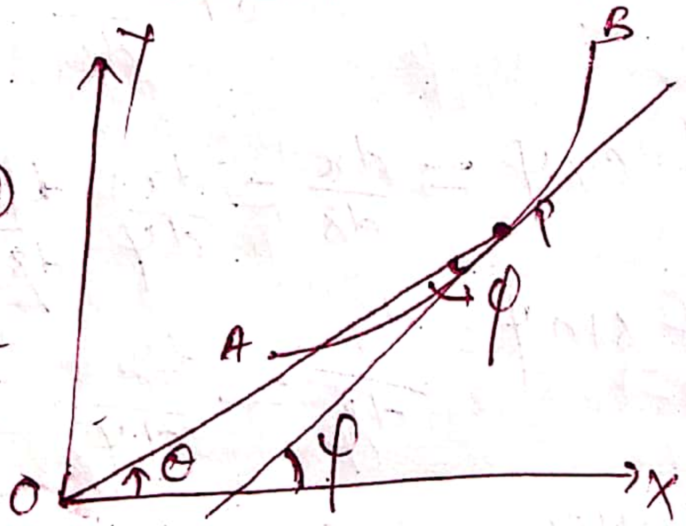
$$= \cos\psi \cdot \frac{d\psi}{dx}$$

$$= \cos\psi \cdot \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$= \cos\psi \cdot \frac{d\psi}{ds} \cdot \frac{1}{\cos\psi} = \frac{d\psi}{ds} = \frac{1}{\rho}.$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{ds}\right) = \frac{1}{\rho}. \quad \text{(Hence proved)}.$$

Pedal formula for radius of curvature:



From figure;

$$\psi = \theta + \phi \quad \text{--- (1)}$$

Differentiating both sides w.r.t 's' we have.

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \Rightarrow \frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\phi}{dr} \cdot \frac{dr}{ds}$$

$$\frac{1}{\rho} = \frac{1}{r}\cdot \sin\phi + \cos\phi \frac{d\phi}{dr}.$$

$$\left(\frac{d\theta}{ds} = \frac{1}{r}\sin\phi ; \ \frac{dr}{ds} = \cos\phi \right)$$

$$\frac{1}{\rho} = \frac{1}{r}\left(\sin\phi + r\cos\phi \cdot \frac{d\phi}{dr} \right).$$

$$= \frac{1}{r}\cdot \frac{d}{dr}(r\sin\phi). \quad (\text{as } p = r\sin\phi)$$

$$= \frac{1}{r}\cdot \frac{dp}{dr}$$

$$\Rightarrow \boxed{\rho = r\frac{dr}{dp}}$$ This is required eqn.

Example :- Find the radius of curvature at the point (p,r) of cardiod. $r^3 = 2ap^2$.

Soln :- The eqn of cardiod $r^3 = 2ap^2$ —— (1).

Differentiating w.r.t. p; we have

$$3r^2\frac{dr}{dp} = 4ap \Rightarrow r\frac{dr}{dp} = \frac{4}{3}\frac{ap}{r}.$$

$$\Rightarrow \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4a}{3r}\sqrt{\frac{r^3}{2a}}$$

$$\rho = \frac{2}{3}\sqrt{2ar}.$$