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Polar formula for Radius of curvature :-

By tangents and normals; we know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{--- (1)}$$

Differentiating (1) w.r.t. r; we get

$$-\frac{2}{p^3}\cdot\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{2}{r^4}\left(\frac{dr}{d\theta}\right)\cdot\frac{d^2r}{d\theta^2}\cdot\frac{d\theta}{dr}$$

$$= -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{2}{r^4}\frac{d^2r}{d\theta^2}$$

$$\Rightarrow \frac{r^5}{p^3}\cdot\frac{dp}{dr} = r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right) \quad \text{--- (11)}$$

We know that
$$\rho = r\frac{dr}{dp}.$$

$$\rho = \frac{r \cdot r^5/p^3}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \quad \text{(using (11))}$$

$$= \frac{r^6/p^3}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{r^6\left[\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta}} \quad \text{(using (1)).}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

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Tangential polar formula for Radius of curvature :- A relation between p and φ for a curve is called its tangential polar equation.

We know that
$$\frac{dr}{ds} = \cos\phi$$

and.
$$\rho = \frac{ds}{d\varphi} = r\frac{dr}{dp}$$

and also
$$\frac{dp}{d\varphi} = \frac{dp}{dr} \cdot \frac{dr}{ds} \cdot \frac{ds}{d\varphi}$$
$$= \frac{dp}{dr} \cdot \cos\phi \cdot r\frac{dr}{dp}$$
$$\Rightarrow \frac{dp}{d\varphi} = r\cos\phi \quad \text{--- (I)}$$

and we know that $p = r\sin\phi$ --- (II).

squaring (I) and (II) and adding, we have
$$p^2 + \left(\frac{dp}{d\varphi}\right)^2 = r^2 \quad \text{--- (III)}$$

Differentiating w.r.t. p on both sides, we get
$$2p + 2\frac{dp}{d\varphi} \cdot \frac{d}{dp}\left(\frac{dp}{d\varphi}\right) = 2r\frac{dr}{dp}$$
$$\Rightarrow p + \frac{dp}{d\varphi} \cdot \frac{d}{d\varphi}\left(\frac{dp}{d\varphi}\right) \cdot \frac{d\varphi}{dp} = r \cdot \frac{dr}{dp}$$
$$\Rightarrow p + \frac{d^2p}{d\varphi^2} = \rho$$

which is required formula.

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Example 8 – Show that the radius of curvature for the curve $r^n = a^n \cos n\theta$ is $\frac{a^n}{(n+1) r^{n-1}}$.

Soln :– We have.

$$r^n = a^n \cos n\theta \quad \text{——— (i)}$$

Taking log of both sides; we have

$$\log r^n = \log\{a^n . \cos n\theta\}$$

$$\Rightarrow n \log r = n \log a + \log \cos n\theta \quad \text{——— (ii)}$$

Differentiating both sides w.r.t. 'θ' we have

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-n \sin n\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = -r \tan n\theta \quad \text{——— (iii)}$$

Again differentiating w.r.t. θ; we have

$$\frac{d^2r}{d\theta^2} = -\frac{dr}{d\theta} \tan n\theta - r . \sec^2 n\theta . n$$

$$\frac{d^2r}{d\theta^2} = r \tan^2 n\theta - nr \sec^2 n\theta \quad (\text{using (iii)}) \quad \text{——— (iv)}$$

We know that

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \quad \text{——— (v)}$$

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using eqn (iii) and (iv) in eqn (v); we have

$$\rho = \frac{\left[r^2 + r^2\tan^2 n\theta\right]^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2\left[\sec^2 n\theta + n\sec^2 n\theta\right]}$$

$$= \frac{r\sec n\theta}{(n+1)} = \frac{r}{(n+1)}\cdot\frac{1}{\cos n\theta}$$

using $\cos n\theta = \frac{r^n}{a^n}$

$$\rho = \frac{r}{(n+1)}\cdot\frac{1}{\frac{r^n}{a^n}} = \frac{a^n}{(n+1)}\cdot\frac{r}{r^n}$$

$$\Rightarrow \rho = \frac{a^n}{(n+1)\,r^{n-1}} \quad \text{(Hence proved)}$$

Exercise: (1) For the curve $r^2 = a^2\cos 2\theta$; find the value of (i) ϕ (ii) p (iii) ρ.

Exercise (2): Show that the pedal equation to the curve $\frac{l}{r} = 1 + e\cos\theta$ is

$$\frac{1}{p^2} = \frac{1}{l^2}\left(\frac{2l}{r} - 1 + e^2\right)$$

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