B.Sc. IInd Year
Paper - IIIrd.
Differential Calculus

Envelope
Lecture - 3

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Theorem:- The envelope of a family of curves which is quadratic in parameter α is given by its discriminant equated to zero.

Proof:- Let $A\alpha^2 + B\alpha + C = 0$ — (1); where A, B, C are functions of x and y; then (1) is quadratic in α and α is a parameter. We have determine to envelope of the family of curve given by eqn (1).

Differentiating (1), partially w.r.to α; we have

$$2A\alpha + B = 0$$

$$\Rightarrow \alpha = -\frac{B}{2A} \quad \text{— (II)}$$

Using value of $\alpha = -\frac{B}{2A}$ in eqn (1), we have

$$A\left(-\frac{B}{2A}\right)^2 + B\cdot\left(-\frac{B}{2A}\right) + C = 0$$

$$\Rightarrow \frac{AB^2}{4A^2} - \frac{B^2}{2A} + C = 0$$

$$\Rightarrow AB^2 - 2AB^2 + 4A^2C = 0$$

$$\Rightarrow -AB^2 + 4A^2C = 0$$

$$\Rightarrow B^2 - 4AC = 0 \quad (\text{Taking } A \neq 0)$$

which is statement of Theorem.
Hence the theorem proved.

Example:- Find the envelope of the family of

$$y = mx + \sqrt{(a^2m^2 + b^2)};$$ where a, b are constant & m is a parameter.

Solution:- Given that

$$y = mx + \sqrt{(a^2m^2+b^2)} \quad \text{—— (i)}$$

squaring both sides; we have

$$y^2 = m^2x^2 + (a^2m^2+b^2) + 2mx\sqrt{(a^2m^2+b^2)}$$

or rearranging eqn (i) $y - mx = \sqrt{a^2m^2+b^2}$

and squaring; we have

$$y^2 + m^2x^2 - 2mxy = a^2m^2+b^2$$

$$\Rightarrow m^2(x^2-a^2) - 2xy \cdot m + (y^2-b^2) = 0 \quad \text{—— (ii)}$$

which is quadratic in m; as a parameter.

So envelope is given by discriminant equated to zero.

$$\Rightarrow (-2xy)^2 - 4(x^2-a^2)(y^2-b^2) = 0$$

Now arranging the terms and simplifying; we obtained

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the required eqn of envelope.

Exercise:- (i) Find the envelope of family of curves given by $y = mx + \frac{1}{m}$.

Exercise (ii):- Find the envelope of family of curves given by $tx^2 + t^2y = a$; where t is a parameter & a is constant