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T.D.C. II (Maths Hons.) Lecture No. 5

Envelopes:-

Let $F(x,y,\alpha)=0$ denotes a family of curves, for α as a parameter (i.e. α varies), then the envelopes of family of curves which touches each member of family of curves.

Alternative definition:- Let $F(x,y,\alpha)=0$ be the given family of curves. Let $F(x,y,\alpha)=0$ and $F(x,y,\alpha+h)=0$ are two consecutives member of family of curves which intersect at a definite point P, as h tends to zero, then the envelope is locus of point of P (as α varies).

Let us consider $F(x,y,\alpha)=0$ —— (1) be the given family of curves.

Let us suppose that $F(x,y,\alpha)=0$ and $F(x,y,\alpha+h)=0$ —— (ii) are two consecutive (neighbouring) member of family of given curves (1).

Since as h tends to zero; $F(x,y,\alpha)=0$ and $F(x,y,\alpha+h)=0$ tends to or intersect at a definite point P. So we have

$$F(x,y,\alpha+h)-F(x,y,\alpha)=0 \quad \text{as } h\to 0$$

$$\Rightarrow \lim_{h\to 0}\frac{F(x,y,\alpha+h)-F(x,y,\alpha)}{h}=0$$

By def^n of partial derivatives, we have

$$\frac{\partial F}{\partial \alpha}=0 \quad \text{—— (iii)}$$

By eliminating α, from the eqns (1) & (iii) we have locus of point P; which is envelope of the eqn (1).

$\Rightarrow$ Combining eqn (1) & (iii) also gives envelope of the family of curves represented by eqn (1).

Example ①:- Find the envelope of the family of curves

$$y = mx + \frac{1}{m} \quad ; \quad m \text{ is a parameter.}$$

Solution:- We have $y = mx + \frac{1}{m}$ —— (I)

$$\Rightarrow F(x, y, m) = y - mx - \frac{1}{m} = 0 \text{ —— (II)}$$

Differentiating partially above eqn w.r.to m; obtained

$$\frac{\partial F}{\partial m} = -x + \frac{1}{m^2} = 0$$

$$\Rightarrow \frac{1}{m^2} = x \Rightarrow m^2 = \frac{1}{x} \Rightarrow m = \pm \frac{1}{\sqrt{x}} \text{ —— (III)}$$

using in eqn (II); we have

$$y - \left(\pm \frac{1}{\sqrt{x}}\right) \cdot x - \frac{1}{\left(\pm \frac{1}{\sqrt{x}}\right)} = 0$$

$$\Rightarrow y \mp \sqrt{x} \mp \sqrt{x} = 0 \Rightarrow y = \pm 2\sqrt{x}$$

squaring both sides, we have $y^2 = 4x$ —— (iv)

which is required envelope for the given family of curves.

Example: ② :- Find the envelope of the family of curves $r^2 = a^2 \cos 2\theta$. where a being parameter.

Solution:- Given that $r^2 = a^2 \cos 2\theta$ —— (I)

Family of curves is given by

$$F(r, \theta, a) = r^2 - a^2 \cos 2\theta = 0 \text{ —— (II)}$$

Partially defferentiating eqn (II), w.r.t. a; we have

$$\frac{\partial F}{\partial a} = 2a \cos 2\theta = 0 \text{ —— (III)}$$

$$\Rightarrow a = 0 \text{ —— (iv)}$$

using eqn (iv) in eqn (II); we have

$$r = 0$$

which is required envelope of the given family of curves $r^2 = a^2 \cos 2\theta$.