To D.C. I

: Part - II (2)

Some Examples on the basis of Taylor's and Maclaurin's series :

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Example (1) :- If $y = \sin(m\sin^{-1}x)$, then show $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$.
Hence or otherwise expand $\sin m\theta$ in powers of $\sin\theta$.

Solution :- Given that $y = \sin(m\sin^{-1}x)$ ——— (i)

Differentiating (i) w.r.to x; we obtained

$$\frac{dy}{dx} = \cos(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}} \qquad \text{——— (ii)}$$

Squaring both sides, we have

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = \left[1-\sin^2(m\sin^{-1}x)\right]\cdot m^2$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = (1-y^2)m^2 \qquad (\text{using eq}^n \text{ (i)}) \text{ (iii)}$$

Again differentiating eq^n (iii) w.r.to x, we obtained

$$2(1-x^2)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -2m^2y\frac{dy}{dx} \qquad \text{——— (iv)}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0 \text{ — (v) Hence the result.}$$

Eq^n (v) differentiating n times; we have

$$\left[\text{using } \frac{dy}{dx} = y_1,\ \frac{d^2y}{dx^2} = y_2, \ldots, \frac{d^ny}{dx^n} = y_n, \ldots\right]$$

$$y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{\lfloor 2}y_n(-2) - xy_{n+1} - ny_n + m^2y_n = 0 \qquad \text{——— (v)}$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0 \qquad \text{——— (vi)}$$

Now, putting $x = 0$, we obtained $(y_{n+2})_{x=0} = (n^2-m^2)(y_n)_{x=0}$ ——— (vii)

Which gives; $y = 0$, $y_1 = m$, $y_2 = 0$, $y_3 = (1-m^2)\cdot m$, $y_4 = 0$ ... etc. at $x=0$ ——— (viii)

By Maclaurin's series; we know that $f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \cdots + \frac{x^n}{\lfloor n}f^n(0) + \cdots$

using (viii), we have with the help of $\sin^{-1}x = \theta$ i.e. $\sin\theta = x$

$$\sin m\theta = m\sin\theta + \frac{1}{\lfloor 3}m(-m^2+1)\sin^3\theta + \frac{\sin^5\theta}{\lfloor 5}m(1-m^2)(3^2-m^2) + \text{——}$$

Example ② :- Expand $\sin x$ in powers of $(x-\frac{\pi}{2})$.

Solution:- Given that $f(x)=\sin x$ — ① then $f(\frac{\pi}{2})=\sin\frac{\pi}{2}=1$.

Differentiating w.r.to x, and using $x=\frac{\pi}{2}$; we have

$$\left.\begin{array}{ll} f'(x)=\cos x & ;\ f'(\frac{\pi}{2})=0 \\ f''(x)=-\sin x & ;\ f''(\frac{\pi}{2})=-1 \\ f'''(x)=-\cos x & ;\ f'''(\frac{\pi}{2})=0 \\ f^{iv}(x)=\sin x & ;\ f^{iv}(\frac{\pi}{2})=1 \end{array}\right\} \text{(II)}$$

and so on.

By Taylor's series, we know that

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\lfloor 2}f''(a)+\cdots+\frac{(x-a)^n}{\lfloor n}f^n(a)+\cdots \quad \text{(III)}$$

using eqn (II) in above; we have; as $f(x)=\sin x$ and $a=\frac{\pi}{2}$

$$\sin x=f(\tfrac{\pi}{2})+(x-\tfrac{\pi}{2})f'(\tfrac{\pi}{2})+\cdots+\cdots$$

$$\sin x=1-\frac{(x-\frac{\pi}{2})^2}{\lfloor 2}+\frac{(x-\frac{\pi}{2})^4}{\lfloor 4}-\cdots$$ is required result.

Exercise ① :- Expand $2x^3+7x^2+x-1$ in powers of $(x-2)$ by Taylor's series.

Exercise ② :- By Maclaurin's series; show that

$$e^x=1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots$$

Exercise ③ :- By Maclaurin's Theorem or otherwise find the expansion of $y=\sin(e^x-1)$ upto and including the terms in x^4.

Exercise ④ :- Expand $\cos x$ in powers of $(x-\frac{\pi}{4})$.