B.Sc. IInd Year
Maths Hons.
Advanced Calculus
(Paper- IIIrd).

Arvind Kumar Yadav
Assto Profo,
Deptt. of Mathematics
Raja Singh College.
Siwan

Indeterminate forms :

The indeterminate form $(\infty - \infty)$:

Let $\lim_{x \to a} \{\phi(x) - \psi(x)\}$;

when $\lim_{x \to a} \phi(x) = \infty$, $\lim_{x \to a} \psi(x) = \infty$.

This can be put in the form of $\frac{0}{0}$ or in form of $\frac{\infty}{\infty}$ as shown below:

$$\lim_{x \to a} [\phi(x) - \psi(x)] = \lim_{x \to a} \left[\frac{1}{\frac{1}{\phi(x)}} - \frac{1}{\frac{1}{\psi(x)}} \right]$$

$$= \lim_{x \to a} \left[\frac{\frac{1}{\psi(x)} - \frac{1}{\phi(x)}}{\frac{1}{\phi(x)} \cdot \frac{1}{\psi(x)}} \right]$$

which is the form of $\frac{0}{0}$.

Example:- Evaluate $\lim_{x \to \pi/2} (\sec x - \tan x)$

Solution: $\lim_{x \to \pi/2} \sec x = \infty$, $\lim_{x \to \pi/2} \tan x = \infty$

Given is form $(\infty - \infty)$.

$$\lim_{x \to \pi/2} (\sec x - \tan x) = \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to \pi/2} \left(\frac{1 - \sin x}{\cos x} \right) \quad \left(\text{Form } \frac{0}{0} \right)$$

$$= \lim_{x \to \pi/2} \left(\frac{-\cos x}{\sin x} \right) \quad \text{(using de-L'Hospital's Rule)}$$

$$= -\frac{0}{1} = 0$$

$$\Rightarrow \lim_{x \to \pi/2} (\sec x - \tan x) = 0.$$

The indeterminate form of 0^0, 1^∞ and ∞^0:

Let $\lim_{x\to a}[\phi(x)]^{\varphi(x)}$.

When ⓐ $\lim_{x\to a}\phi(x)=0$; $\lim_{x\to a}\varphi(x)=0$

ⓑ $\lim_{x\to a}\phi(x)=1$; $\lim_{x\to a}\varphi(x)=\infty$

ⓒ $\lim_{x\to a}\phi(x)=\infty$; $\lim_{x\to a}\varphi(x)=0$.

Taking $y=[\phi(x)]^{\varphi(x)}$.

Taking natural log of both sides, we have

$$\log y=\log[\phi(x)]^{\varphi(x)}$$

$$=\varphi(x).\log\{\phi(x)\} \quad \text{(Property of log)}$$

$$\Rightarrow \lim_{x\to a}\log y=\lim_{x\to a}[\varphi(x).\log\{\phi(x)\}]$$

Which gives the case arises, study in previous lecture the form of $0\times\infty$ are solved.

$$\Rightarrow \lim_{x\to a}\log y=A$$

$$\Rightarrow \lim_{x\to a}y=e^A \quad \text{(Taking anti-log both sides)}$$

$$\Rightarrow \lim_{x\to a}\{\phi(x)\}^{\varphi(x)}=e^A.$$

Example ①:- Evaluate $\lim_{x \to 0} x^x$.

Solution:- Let $y = x^x$

Taking $\log_e$, both sides, we have

$$\log y = \log x^x = x \log x$$

Now,

$$\lim_{x \to 0} \log y = \lim_{x \to 0} x \cdot \log x \quad \left(\text{form } 0 \times \infty\right)$$

$$= \lim_{x \to 0} \left(\frac{\log x}{\frac{1}{x}}\right) \quad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \quad \left(\text{using del'Hospital's Rule}\right)$$

$$= \lim_{x \to 0} - \frac{x^2}{x} = \lim_{x \to 0} -x$$

$$\Rightarrow \lim_{x \to 0} \log y = 0$$

Taking anti-log both sides, we have

$$\lim_{x \to 0} y = e^0 = 1$$

$$\Rightarrow \lim_{x \to 0} x^x = 1.$$

Example ②:- Evaluate $\lim_{x \to 0} (\cos x)^{\cot x}$.

Solution:- $\lim_{x \to 0} \cos x = 1$; $\lim_{x \to 0} \cot x = \infty$

This is form of 1^∞.

Let $y = (\cos x)^{\cot x}$.

Taking $\log_e$ both sides, we have

$$\log y = \cot x \cdot \log \cos x$$

$$\Rightarrow \lim_{x\to 0} \log y = \lim_{x\to 0} \{\cot x \cdot \log \cos x\}$$

(Form $\infty \times 0$)

$$= \lim_{x\to 0} \frac{\log \cos x}{\tan x} \quad (\text{Form } \tfrac{0}{0})$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{\sec^2 x}$$

$$= \lim_{x\to 0} \frac{-\tan x}{\sec^2 x}$$

$$= \lim_{x\to 0} (-\sin x \cdot \cos x)$$

$$\Rightarrow \lim_{x\to 0} \log y = 0$$

Taking anti-log both sides, we have

$$\lim_{x\to 0} y = e^0$$

$$\Rightarrow \lim_{x\to 0} (\cos x)^{\cot x} = e^0 = 1.$$

Exercise :— Evaluate.

(i) $\lim_{x\to 0} \frac{\log x}{\cos x}$ (ii) $\lim_{x\to 0} x \log \sin x$.

(iii) $\lim_{x\to \infty} \frac{\log x}{x}$ (iv) $\lim_{x\to \pi/2} (\sec x)^{\cot x}$.