B.Sc. IInd Year
Maths Hons
Advanced Calculus
(Paper- IIIrd).

Arvind Kumar Yadav
Asstt Prof, Deptt of Mathematics
Raja Singh College Siwan

The indeterminate form $\frac{\infty}{\infty}$:-

Proposition:-

If $\lim_{x\to a} \phi(x) = \infty$ and $\lim_{x\to a} \varphi(x) = \infty$, then

$$\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\varphi'(x)}.$$

Proof:- We know that

$$\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\left\{\frac{1}{\varphi(x)}\right\}}{\left\{\frac{1}{\phi(x)}\right\}} \quad \text{— (i)}$$

as $\lim_{x\to a} \phi(x) = \infty \Rightarrow \lim_{x\to a} \frac{1}{\phi(x)} = 0$

and $\lim_{x\to a} \varphi(x) = \infty \Rightarrow \lim_{x\to a} \frac{1}{\varphi(x)} = 0$

$$\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\frac{1}{\varphi(x)}}{\frac{1}{\phi(x)}} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to a} \frac{-\left\{\frac{1}{\varphi(x)}\right\}^2 \cdot \varphi'(x)}{-\left\{\frac{1}{\phi(x)}\right\}^2 \cdot \phi'(x)}$$

$$\Rightarrow \lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to a} \left\{\frac{\varphi'(x)}{\phi'(x)}\right\}\left\{\frac{\phi(x)}{\varphi(x)}\right\}^2 \quad \text{— (ii)}$$

Let us suppose that $\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = l$ — (iii)

Case I:- Let $\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = l, \neq 0, \neq \infty$

using in eqn (ii)

$$\Rightarrow l = \lim_{x\to a} \left\{\frac{\varphi'(x)}{\phi'(x)}\right\} \cdot l^2$$

$$\Rightarrow \lim_{x\to a} \frac{\phi'(x)}{\varphi'(x)} = l$$

$$\Rightarrow \lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\varphi'(x)}.$$

Case II : Let $\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = l = 0.$

In eqn (ii) ; adding 1 both side ; we have

$$\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} + 1 = l+1$$

$$\Rightarrow \lim_{x\to a} \frac{\phi(x)+\varphi(x)}{\varphi(x)} = l+1, \neq 0, \neq \infty.$$

So using case I ; we have

$$\Rightarrow \lim_{x\to a} \frac{\phi(x)+\varphi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\phi'(x)+\varphi'(x)}{\varphi'(x)}$$

$$\Rightarrow \lim_{x\to a} \frac{\phi(x)}{\varphi(x)} + 1 = \lim_{x\to a} \frac{\phi'(x)}{\varphi'(x)} + 1$$

$$\Rightarrow \lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\varphi'(x)}.$$

Case III : Let $\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \infty.$

$$\Rightarrow \frac{1}{\lim_{x\to a} \frac{\phi(x)}{\varphi(x)}} = \lim_{x\to a} \frac{\varphi(x)}{\phi(x)} \quad \text{(As Case II)}$$

$$= \lim_{x\to a} \frac{\varphi'(x)}{\phi'(x)}$$

$$\Rightarrow \lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\varphi'(x)}.$$

In all case it happened that $\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\varphi'(x)}$ for form $\frac{\infty}{\infty}$.

The form $\frac{\infty}{\infty}$ is generally changed to the form $\frac{0}{0}$ at some convenient stage.

Example ①:– Evaluate $\lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x}$.

Solution:– Here $\phi(x) = \log(1-x)$

$$\varphi(x) = \cot \pi x$$

$$\lim_{x\to 1} \phi(x) = \lim_{x\to 1} \log(1-x) = \log 0 = -\infty.$$

and $\lim_{x\to 1} \varphi(x) = \lim_{x\to 1} \cot \pi x = \cot \pi$

$$= \infty \qquad (\text{as } \tan\pi = 0)$$

So $\lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x}$ is form $\frac{\infty}{\infty}$.

$$\Rightarrow \lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x} = \lim_{x\to 1} \frac{\frac{1}{(1-x)}\cdot(-1)}{-\operatorname{cosec}^2 \pi x \cdot (\pi)}$$

$$= \lim_{x\to 1} \frac{-\left(\frac{1}{1-x}\right)}{-\pi \operatorname{cosec}^2 \pi x} \quad \text{form } \frac{\infty}{\infty}.$$

$$= \lim_{x\to 1} \frac{\sin^2 \pi x}{\pi(1-x)} \quad \left(\begin{array}{l}\text{change}\\ \text{into form } \frac{0}{0}\end{array}\right)$$

$$= \lim_{x\to 1} \frac{2\sin \pi x \cdot \cos \pi x \cdot \pi}{\pi(-1)}$$

$$= \lim_{x\to 1} (-\sin 2\pi x)$$

$$= -\sin 2\pi = 0$$

Hence $\lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x} = 0$.

The form $0 \times \infty$:

$\lim_{x \to a} \phi(x) \cdot \varphi(x)$; when $\lim_{x \to a} \phi(x) = 0$

and $\lim_{x \to a} \varphi(x) = \infty$

This can be reduced to either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form as shown below:

$$\lim_{x \to a} \phi(x) \cdot \varphi(x) = \lim_{x \to a} \frac{\phi(x)}{\left(\frac{1}{\varphi(x)}\right)}.$$

$$\text{or } \lim_{x \to a} \phi(x) \cdot \varphi(x) = \lim_{x \to a} \frac{\varphi(x)}{\left(\frac{1}{\phi(x)}\right)}$$

respectively.

Example: Evaluate $\lim_{x \to 0} x \log \sin x$.

Solution: Here $\lim_{x \to 0} \phi(x) = \lim_{x \to 0} x = 0$

& $\lim_{x \to 0} \log \sin x = \log \sin 0 = \log 0 = -\infty$.

$$\lim_{x \to 0} x \cdot \log \sin x = \lim_{x \to 0} \frac{\log \sin x}{\left(\frac{1}{x}\right)}; \text{ form } \frac{\infty}{\infty}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} \cdot \cos x}{\left(-\frac{1}{x^2}\right)}.$$

$$= \lim_{x \to 0} \frac{-x^2 \cos x}{\sin x}, \text{ form } \frac{0}{0}$$

$$= \lim_{x \to 0} -\left\{\frac{2x \cos x - x^2 \sin x}{\cos x}\right\}$$

$$= \frac{0}{1} = 0$$

$\Rightarrow \lim_{x \to 0} x \log \sin x = 0$.