Arvind Kr. Yadav
Asstt. Prof.
Deptt. of Mathematics
Raja Singh College Siwan

Advanced Calculus of functions of a single variable :-

Lecture - I

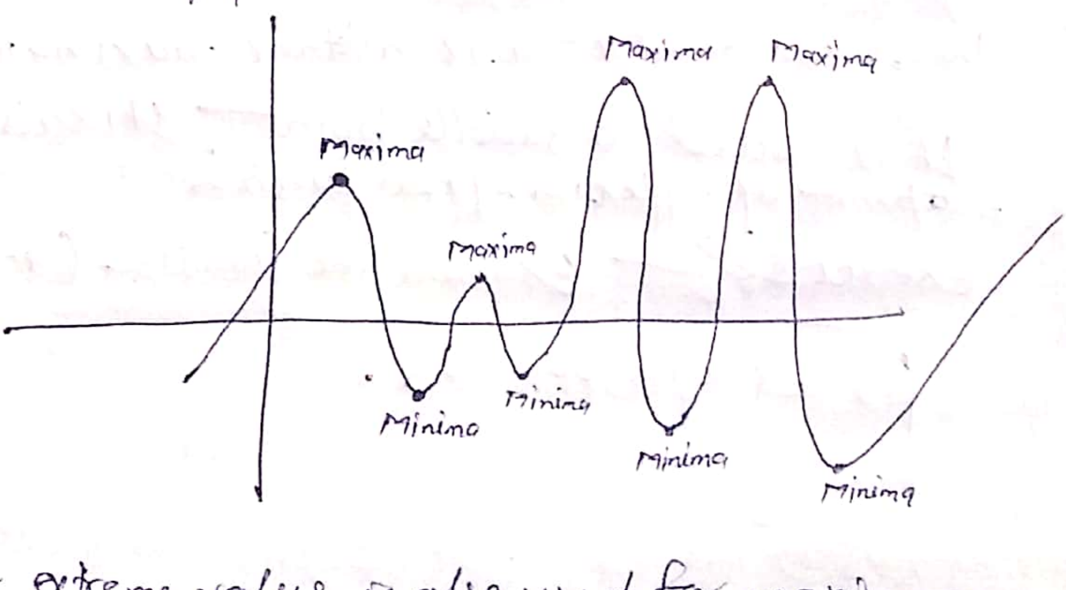
Let $f(x)$ be a function of a single variable x. Suppose that $f(x)$ is a continuous function in the neighbourhood of a point $x=a$. The value $f(a)$ is said to be maximum

if $f(a+h) < f(a)$, $f(a-h) < f(a)$ $\forall h > 0$, h is infinitely very small.

Similarly, the value $f(a)$ is said to be minimum

if $f(a+h) > f(a)$; $f(a-h) > f(a)$; $\forall h > 0$; h is infinitely very small.

In other words: a function $f(x)$ is said to have a maximum or minimum at $x=a$ according as $f(a)$ is greater than or less than any other value of the function at any point in the small neighbourhood of $x=a$.



The term extreme value is also used for maximum or minimum value.

Necessary and sufficient conditions for maxima & minima:

Let $f(x)$ be a function of x. Suppose that $f(x)$ is expandible in the neighbourhood of $x=a$ by Taylor's theorem. Then we have

$$f(a+h) = f(a) + \frac{h}{\lfloor 1} f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + \frac{h^n}{\lfloor n} f^n(a+\theta h); \quad 0<\theta<1$$

$$\Rightarrow f(a+h) - f(a) = h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + \frac{h^n}{\lfloor n} f^n(a+\theta h) \quad \text{--- (1)}$$

Similarly;

$$f(a-h) - f(a) = -h f'(a) + \frac{h^2}{\lfloor 2} f''(a) - \cdots + (-1)^{n-1}\frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + (-1)^n \frac{h^n}{\lfloor n} f^n(a+\theta h) \quad \text{--- (II)}$$

From definition, for maximum or minimum at $x=a$, $f(a+h)-f(a)$; $f(a-h)-f(a)$ must have the same sign (-ve for maximum and +ve for minimum).

Since h is small, the sign of R.H.S. of (I) and (II) is governed by their first terms. Obviously the Ist terms of (I) and (II) are of opposite signs. Since for maxima and minima at $x=a$, (I) & (II) must have the same sign, so the coefficient of h, i.e. $f'(a)$ must vanish. i.e. $f'(a)=0$.

Thus for necessary condition that $f(a)$ should have a maximum or minimum at $x=a$ is that $f'(a)=0$. —— (iii)

Now if $f'(a)=0$; then

$$f(a+h)-f(a)=\frac{h^2}{\lfloor 2}f''(a)+\frac{h^3}{\lfloor 3}f'''(a)+ \cdots$$
$$+\frac{h^n}{\lfloor n}f^n(a+\theta h) \quad \text{—— (iv)}$$

and $f(a-h)-f(a)=\frac{h^2}{\lfloor 2}f''(a)-\frac{h^3}{\lfloor 3}f'''(a)+ \cdots$
$$+(-1)^n\frac{h^n}{\lfloor n}f^n(a-\theta h) \quad \text{—— (v)}$$

obviously, the sign of (iv) & (v) are governed by $f''(a)$ for h^2 is always +ve. Thus the sign of $f(a+h)-f(a)$; $f(a-h)-f(a)$ will be the same as that of $f''(a)$.

When $f''(a)$ is -ve then $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are -ve and so there is a maximum at $x=a$.

Hence $f(x)$ has maximum at $x=a$ if $f'(a)=0$ and $f''(a)<0$.

Similarly, when $f''(a)$ is +ve, then $f(a+h)-f(a)$; $f(a-h)-f(a)$ are +ve and so there is a minimum at $x=a$.

Hence $f(x)$ has minimum at $x=a$ if $f'(a)=0$ and $f''(a)>0$.

These are sufficient conditions for maxima & Minima.

Note :- In case if $f''(a)=0$; then we have

$$f(a+h)-f(a)=\frac{h^3}{\lfloor 3}f'''(a)+\frac{h^4}{\lfloor 4}f^{(iv)}(a)+\cdots \quad \text{(VI)}$$

and $$f(a-h)-f(a)=-\frac{h^3}{\lfloor 3}f'''(a)+\frac{h^4}{\lfloor 4}f^{(iv)}(a)+\cdots \quad \text{(VII)}$$

Obviously (VI) & (VII) have the same sign if $f'''(a)=0$ and then the sign is governed by $f^{iv}(a)$. Thus there will be a maximum at $x=a$ if $f^{(iv)}(a)<0$ and minimum if $f^{(iv)}(a)>0$.

In general, if $f'(a)=f''(a)=\cdots=f^{(n-1)}(a)=0$ and $f^{n}(a)\neq 0$ then n must be an even integer for maximum or minimum at $x=a$. There will be a maximum if $f^{n}(a)<0$ and a minimum if $f^{n}(a)>0$.

In case $f^{n}(a)$ is different from zero is the first non-zero derivative and n is an odd integer then the point $x=a$ is neither maxima nor minima. It is called a saddle point. It is also called a point of inflexion of the function.

Exercise :- Examine the function $(x-3)^5(x+1)^4$ for extreme values.