B.Sc. 2nd Year
Advanced Calculus
Paper IIIrd
Lecture - 2

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Properties of Maxima & Minima:

Let $f(x)$ be a continuous function of x. Then

(i) If $f(x)$ is not a constant function then between two equal values of $f(x)$; there must be at least one maximum or minimum.

(ii) Maxima and minima occur alternately.

(iii) A function might have several maxima & minima.

(iv) A function $f(x)$ is maximum at $x=a$ if $f'(x)$ changes sign from +ve to −ve as passes through a from left to right.

(v) A function $f(x)$ is minimum at $x=a$ if $f'(x)$ changes sign from −ve to +ve as x passes through a from left to right.

(vi) If the sign of $f'(x)$ does not change while x passes through a; $f(x)$ has neither maximum nor minimum at $x=a$.

Working Rule for Maxima & Minima of fun $f(x)$:

Step I: Let $f(x)$ be the given function.

Step II: Find $f'(x)$ and equate to zero i.e. $f'(x)=0$

Step III: Find values of x from $f'(x)$; say $x=\alpha, \beta, \gamma \ldots$ etc.

Step IV: Find $f''(x)$; determine $f''(\alpha)$; if −ve maxima; if +ve minima exist; if $f''(\alpha)$ is zero further check. Proceed same way for other values of x.

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Maxima and Minima of functions of two independent variables:- Let $f(x,y)$ be a function of two independent variables x and y. and let $f(x,y)$ be continuous for all values of x and y in the small neighbourhood of $x=a$; $y=b$.

Then $f(x,y)$ is said to have a maximum or minimum at (a,b) according as $f(a+h, b+k)$ is less than or greater than $f(a,b)$ for all sufficiently small independent values of h and k in the small neighbourhood of (a,b).

Necessary conditions for the existence of a maximum and minimum:- Let $f(x,y)$ be a function of two independent variables x and y. By Taylor's theorem for two variables we have

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \cdots$$

$$+ \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots \quad (1)$$

By taking h and k sufficiently small, the first degree terms in h and k can be made to govern the sign of right-hand side and therefore left hand side of (1). Now h and k are both +ve as well as -ve, thus on changing the sign of h and k the sign of right-hand side and therefore of left hand side changes. But from the above definition for all (+ve and -ve) arbitrary small values of h and k.

$f(x+h, y+k) - f(x,y)$ must be of

invariable sign for the existence of a maximum or minimum value. Thus the necessary condition for maxima & or minima is that first degree terms in (1) must vanish i.e.

$$h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}=0 \quad \text{--------- (II)}$$

Since h and k are independent and non-zero therefore (II) implies that $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

Hence the necessary conditions that $f(x,y)$ should have a maximum or a minimum at $x=a;\ y=b$. are that

$$\frac{\partial f}{\partial x}=0 \text{ and } \frac{\partial f}{\partial y}=0 \text{ at } x=a;\ y=b.$$

These are necessary condition for existence of maxima and minima.

Sufficient condition for a maxima and minima (Lagrange' condition for two Independent variables):

Let $\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}=r$, $\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{(a,b)}=s$

and $\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}=t$.

If the necessary conditions for maximum and minimum at $x=a,\ y=b$ are satisfied, then

$$f(a+h,\ b+k)-f(a,b)=\frac{1}{2}\left[rh^2+2shk+tk^2\right]+R_3$$

where R_3 consists of terms of IIIrd and higher orders in h and k. For sufficiently small values of h & k the signs of R.H.S. and hence of L.H.S. is / page 4 governed by $rh^2 + 2shk + tk^2 = \frac{1}{r}[r^2h^2 + 2rshk + rtk^2]$

$$= \frac{1}{r}\{(rh+sk)^2 + (rt-s^2)k^2\}$$

Clearly, if $rt-s^2$ is +ve, then the whole expression within the square brackets will be governed by r.

In case $(rt-s^2)$ is not +ve then we can say nothing about the sign of this expression.

Hence Lagrange's condition for a minimum is that $(rt-s^2) > 0$ and $r > 0$.

Similarly, Lagrange's condition for a maximum is that $(rt-s^2) > 0$ and $r < 0$.

If $(rt-s^2) < 0$, then we have neither maximum nor minimum for we can say nothing about the sign of the 2nd degree terms in (1).

On the other hand, if $(rt-s^2) = 0$ i.e. $rt = s^2$, we have $rh^2 + 2shk + rk^2 = \frac{1}{r}(rh + ks)^2$ and so it has the same sign as r. Hence it is maximum if $r < 0$ and minimum if $r > 0$.

In case $rt = s^2$ and $rh + ks = 0$ we have $\frac{h}{k} = -\frac{s}{r} = \beta$ (say)

Then second degree terms vanish and we have to consider terms of higher order in (1).

The cubic terms must vanish when $\frac{h}{k} = \beta$ otherwise or changing the signs of h and k we could change the sign of (1).

Exercise :- Prove that $u = x^2y^2 - 5x^2 - 8xy - 5y^2$ is maximum at $x = 0, y = 0$.