B.Sc. IInd Year
Advanced Calculus
Paper IIIrd
Lecture - 1

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Lagrange's condition for three independent variables:

Let $f(x,y,z)$ be a function of three independent variables. We wish to determine sufficient conditions so that f may have a maximum or minimum at (a,b,c).

Let $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial y^2}$, $C = \frac{\partial^2 f}{\partial z^2}$

$F = \frac{\partial^2 f}{\partial y \partial z}$, $G = \frac{\partial^2 f}{\partial z \partial x}$, $H = \frac{\partial^2 f}{\partial x \partial y}$

Since for the existence of maxima and minima at (a,b,c) we must have $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$, therefore

$$f(a+h, b+k, c+l) - f(a,b,c)$$
$$= \frac{1}{\lfloor 2}\left[Ah^2 + Bk^2 + Cl^2 + 2Fkl + 2Glh + 2Hhk\right] + R_3 \quad (1)$$

where R_3 consists of terms of higher order of small quantities h, k, l.

Since h, k, l are sufficiently small and finite so in (1) second degree terms can be made to govern the sign of R.H.S. of (1) and so of L.H.S. of (1).

If the sign be negative, f has a maximum at (a,b,c) and if it be +ve, f is minimum.

Now the IInd degree terms in (1) may be written as

$$I = Ah^2 + Bk^2 + Cl^2 + 2Fkl + 2Glh + 2Hhk.$$

$$= \frac{1}{A}\left[A^2h^2 + ABk^2 + ACl^2 + 2FAkl + 2AGlh + 2AHhk\right]$$

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$$I = \frac{1}{A}\left\{A^2h^2 + 2Ah(Gl+Hk) + ABk^2 + ACl^2 + 2AFkl\right\}$$

$$= \frac{1}{A}\left[\left\{Ah + (Gl+Hk)\right\}^2 + ABk^2 + ACl^2 + 2AFkl - (Gl+Hk)^2\right]$$

$$= \frac{1}{A}\left[(Ah+Hk+Gl)^2 + (AB-H^2)k^2 + 2lk(AF-GH) + (AC-G^2)l^2\right] \quad \text{--- (11)}$$

Now the sign of I will be the same as that of A if the sign of the expression with the brackets of (11) is +ve. For this $AB-H^2 > 0$

and $(AB-H^2)(AC-G^2) - (AF-GH)^2 > 0$

Thus I will be +ve if

$A>0$, $AB-H^2>0$, and $ABC+2FGH-AF^2-BG^2-CH^2>0$

Hence the function $f(x,y,z)$ will be minimum at (a,b,c) if

$$A>0, \quad \begin{vmatrix} A & H \\ H & B \end{vmatrix} > 0 \text{ and } \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} > 0$$

and f will be maximum at (a,b,c) if above three expression are alternatively -ve and +ve. i.e.

$$A<0; \quad \begin{vmatrix} A & H \\ H & B \end{vmatrix} > 0 \text{ and } \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} < 0$$

But, if these conditions are not satisfied then we can say that nothing about the sign of second degree terms in (1) and so in that case $f(x,y,z)$ will be neither maximum nor minimum at (a,b,c).

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Example: Prove that $u=(x+y+z)^3-3(x+y+z)-24xyz+a^3$ has a minimum at $(1,1,1)$ and maximum at $(-1,-1,-1)$.

Solution: Given that

$$u=(x+y+z)^3-3(x+y+z)-24xyz+a^3 \quad \text{--- (I)}$$

For maxima and minima; we have

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=3(x+y+z)^2-3-24yz=0 \quad \text{--- (II)}\\ \frac{\partial u}{\partial y}&=3(x+y+z)^2-3-24xz=0 \quad \text{--- (III)}\\ \frac{\partial u}{\partial z}&=3(x+y+z)^2-3-24xy=0 \quad \text{--- (IV)}\end{aligned}\right\}$$

Equations (II), (III) & (IV); gives

$$(x+y+z)^2-1=8yz=8xz=8xy$$

$$\Rightarrow yz=xz=xy.$$

$$\Rightarrow \frac{1}{x}=\frac{1}{y}=\frac{1}{z} \quad \text{or } x=y=z$$

Then $(x+y+z)^2-1=8yz$ gives

$$9x^2-1=8x^2 \Rightarrow x=\pm 1$$

Thus $x=y=z=1$ or $x=y=z=-1$

Now, $A=\dfrac{\partial^2 u}{\partial x^2}=6(x+y+z) = \begin{cases} 18 \text{ at } (1,1,1) \\ -18 \text{ at } (-1,-1,-1)\end{cases}$

$B=\dfrac{\partial^2 u}{\partial y^2}=6(x+y+z)= \begin{cases} 18 \text{ at } (1,1,1) \\ -18 \text{ at } (-1,-1,-1)\end{cases}$

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$$C = \frac{\partial^2 u}{\partial z^2} = 6(x+y+z) = \begin{cases} 18 & \text{at } (1,1,1) \\ -18 & \text{at } (-1,-1,-1) \end{cases}$$

$$F = \frac{\partial^2 u}{\partial y \partial z} = 6(x+y+z) - 24x = \begin{cases} -6 & \text{at } (1,1,1) \\ 6 & \text{at } (-1,-1,-1) \end{cases}$$

Similarly; $G = H = \begin{cases} -6 & \text{at } (1,1,1) \\ 6 & \text{at } (-1,-1,-1) \end{cases}$

Thus at the point $(1,1,1)$, we have

$A = 18$ (+ve) ; $\begin{vmatrix} A & H \\ H & B \end{vmatrix} = 288$ (+ve)

and $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 3456$ (+ve)

Hence u is minimum at $(1,1,1)$.

Again at the point $(-1,-1,-1)$; we have

$A = -18$ (-ve) ; $\begin{vmatrix} A & H \\ H & B \end{vmatrix} = 288$ (+ve)

and $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = -3456$ (-ve)

Thus u is maximum at $(-1,-1,-1)$.

Exercise: Show that when $x = y = z = 1$ the function $f(x,y,z) = x^4 + y^4 + z^4 - 4xyz$ has a stationary value and investigate its nature.

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