Arvind Kr. Yadav
Asstt. Prof.
Department of Mathematics
Raja Singh College
Siwan.

B.Sc. IInd Year
Advanced Calculus
Paper-IIIrd
Lecture-5

Lagrange's method of undetermined multipliers:

Let $u = \phi(x_1, x_2, \ldots, x_n)$ be a function of n variables $x_1, x_2, \ldots x_n$. Let these variables connected by following r equation

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_r(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}\right\} \qquad \text{(i)}$$

Thus there are only $(n-r)$ independent variables out of these n variables.

For extreme of u; we have

$$du = \frac{\partial u}{\partial x_1}\cdot dx_1 + \frac{\partial u}{\partial x_2}\cdot dx_2 + \cdots + \frac{\partial u}{\partial x_n}\cdot dx_n = 0 \qquad \text{(ii)}$$

Also
$$df_1 = \frac{\partial f_1}{\partial x_1}\cdot dx_1 + \frac{\partial f_1}{\partial x_2}\cdot dx_2 + \cdots + \frac{\partial f_1}{\partial x_n}\cdot dx_n = 0 \qquad \text{(iii)}$$

$$df_2 = \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2}\cdot dx_2 + \cdots + \frac{\partial f_2}{\partial x_n}\cdot dx_n = 0 \qquad \text{(iv)}$$

$$\vdots$$

$$df_r = \frac{\partial f_r}{\partial x_1}\cdot dx_1 + \frac{\partial f_r}{\partial x_2}\cdot dx_2 + \cdots + \frac{\partial f_r}{\partial x_n}\cdot dx_n = 0 \qquad \text{(v)}$$

Multiplying equations (ii), (iii), (iv), ... (v) by $1, \lambda_1, \lambda_2, \ldots, \lambda_r$ respectively and adding, we get

$$A_1 dx_1 + A_2 dx_2 + \cdots + A_n dx_n = 0 \qquad \text{(vi)}$$

where $A_i = \frac{\partial u}{\partial x_i} + \lambda_1 \frac{\partial f_1}{\partial x_i} + \lambda_2 \frac{\partial f_2}{\partial x_i} + \cdots + \lambda_r \frac{\partial f_r}{\partial x_i}$ and $i = 1, 2, \ldots, n$

Now, we have at our choice the r quantities $\lambda_1, \lambda_2, \ldots, \lambda_r$ and we choose them in such a way that

$$A_1 = 0 = A_2 = A_3 = \cdots = A_r \qquad \text{(VII)}$$

Then the equation (VI) is reduced to

$$A_{r+1}\, dx_{r+1} + A_{r+2}\, dx_{r+2} + \cdots + A_n\, dx_n = 0 \qquad \text{(VIII)}$$

Let us now suppose that out of n variables the following (n-r) variables are independent:

$$x_{r+1}, x_{r+2}, \ldots, x_n.$$

Then as $dx_{r+1}, dx_{r+2}, \ldots, dx_n$ are independent, so their coefficient are separately zero and we have

$$A_{r+1} = A_{r+2} = \cdots = A_n = 0 \qquad \text{(IX)}$$

$\therefore$ From (VII) and (IX), we have

$$A_1 = 0 = A_2 = \cdots = A_r = A_{r+1} = \cdots = A_n \qquad \text{(X)}$$

Also from (I), we have

$$f_1 = 0 = f_2 = \cdots = f_r \qquad \text{(XI)}$$

From (X) and (XI) we get (n+r) equations which determine the r multipliers $\lambda_1, \lambda_2, \ldots, \lambda_r$ and get the possible values of u.

Example :- Prove that the rectangular solid of maximum volume that can be inscribed in a sphere is cube.

Solution :- Let the radius of sphere be a and the co-ordinate of an angular point of the rectangular solid in the +ve octant be x, y, z. Then it is clear that the length of edges of inscribed rectangular solid be $2x, 2y, 2z$. Therefore, the volume V of the rectangular solid is given by $V = 2x \cdot 2y \cdot 2z = 8xyz$ —— (I)

Since the angular points lies on the sphere,

$$x^2 + y^2 + z^2 = a^2 \text{ —— (II)}$$

Here we have to discuss the maxima of V under condition (II). We now define an auxilliary function

$F(x,y,z)$, where $F(x,y,z) = 8xyz + \lambda(x^2+y^2+z^2-a^2)$

For maxima or minima of the function,

$$\frac{\partial F}{\partial x} = 8yz + 2\lambda x = 0 \text{ —— (III)}$$

$$\frac{\partial F}{\partial y} = 8zx + 2\lambda y = 0 \text{ —— (iv)}$$

$$\& \quad \frac{\partial F}{\partial z} = 8xy + 2\lambda z = 0 \text{ —— (V)}$$

$$\therefore \frac{x}{yz} = \frac{y}{zx} = \frac{z}{xy} = -\frac{4}{\lambda}$$

$$\Rightarrow x^2 = y^2 = z^2 = -4$$

$$\Rightarrow \frac{x^2}{1} = \frac{y^2}{1} = \frac{z^2}{1}$$

$$\Rightarrow \frac{x^2}{1} = \frac{y^2}{1} = \frac{z^2}{1} = \frac{x^2+y^2+z^2}{1+1+1} = \frac{a^2}{3} \quad (\text{From eqn (II)})$$

$$\Rightarrow x = y = z = \frac{9}{\sqrt{3}} \; ; \; \& \lambda = -\frac{49}{\sqrt{3}} \text{ —— (VI)}$$

Now $d^2F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)^2 F$

$$= 2\lambda(dx^2 + dy^2 + dz^2) + 16(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

$$= -\frac{8a}{\sqrt{3}}\left\{dx^2 + dy^2 + dz^2 - 2(dx\,dy + dy\,dz + dz\,dx)\right\}$$

Also from eqn (iv); we have

$$2x\,dx + 2y\,dy + 2z\,dz = 0$$

$$\Rightarrow dx + dy + dz = 0 \quad (\text{using eqn (vi)})$$

so $d^2F = -\frac{8a}{\sqrt{3}}\left[(dx+dy)^2 + dz^2 - 2dz(dx+dy)\right]$

$$= -\frac{8a}{\sqrt{3}}\left[(dx+dy)^2 + dz^2 + 2dz^2\right] < 0$$

Hence for $x = y = z = \frac{a}{\sqrt{3}}$; the function $F(x,y,z)$ and so V is maximum.

Therefore, the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Also from the equation (vi), the maximum value of V is given by

$$V_{max.} = (8xyz)_{at\ x=y=z=\left(\frac{a}{\sqrt{3}}\right)} = 8\cdot\frac{a}{\sqrt{3}}\cdot\frac{a}{\sqrt{3}}\cdot\frac{a}{\sqrt{3}} = \frac{8a^3}{3\sqrt{3}}$$

Exercise:- Show that if the perimeter of a triangle is constant, its area is maximum when it is equilateral.