B.Sc. Ist Year
Maths. Hons.
(B.Sc. IInd Year Maths Subs.)
Diff. Calculus (Paper IInd).

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# Tangents & Normals :

We know that angle between radius vector $r$ and tangent in polar form is given by

$$\tan\phi = r\frac{d\theta}{dr} \quad \text{—— (I)}$$

$$\cos\phi = \frac{1}{\sec\phi} = \frac{1}{\sqrt{\sec^2\phi}} = \frac{1}{\sqrt{1+\tan^2\phi}}$$

$$= \frac{1}{\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}} = \frac{dr/d\theta}{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}}$$

$$\cos\phi = \frac{dr/d\theta}{ds/d\theta} \;; \quad \frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$\Rightarrow \cos\phi = \frac{dr}{ds}. \quad \text{—— (II)}$$

Again
$$\sin\phi = \frac{1}{\operatorname{cosec}\phi} = \frac{1}{\sqrt{\operatorname{cosec}^2\phi}} = \frac{1}{\sqrt{1+\cot^2\phi}}$$

$$= \frac{\tan\phi}{\sqrt{1+\tan^2\phi}} = r\frac{d\theta}{dr}\cdot\cos\phi$$

$$= r\cdot\frac{d\theta}{dr}\cdot\frac{dr}{ds}$$

$$\Rightarrow \sin\phi = r\frac{d\theta}{ds} \quad \text{—— (III)}$$

**Example :** Prove that
$$\sin^2\phi\frac{d\phi}{d\theta} + r\frac{d^2r}{ds^2} = 0.$$

**Solution :** We know that $\cos\phi = \frac{dr}{ds}$

$$\sin\phi = r\frac{d\theta}{ds}.$$

So we have.

$$\frac{d^2r}{ds^2} = -\sin\phi\frac{d\phi}{ds}$$

$$\Rightarrow \sin\phi\frac{d\phi}{ds} + \frac{d^2r}{ds^2} = 0$$

$$\Rightarrow r\sin\phi\frac{d\phi}{ds} + r\frac{d^2r}{ds^2} = 0.$$

$$\Rightarrow \sin^2\phi\frac{d\phi}{d\theta} + r\frac{d^2r}{ds^2} = 0 \left(\text{using } \sin\phi \text{ for } r\frac{d\theta}{ds}\right)$$

(Hence proved).

**Pedal Equation:** A relation between $p$ & $r$ for a given curve is called its pedal eqn.

**Find the pedal eqn from cartesian equation:**

Let the eqn of curve be $y = f(x)$. The tangent to the curve at $(x, y)$ is given by

$$Y - y = \left(\frac{dy}{dx}\right)(X - x) \quad \text{(I)}$$

Thus the length $p$ of the $\perp^{r}$ on its from origin $(X=0, Y=0)$ is given by

$$p = \frac{x\frac{dy}{dx} - y}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \quad \text{(II)}$$

& we have $r^2 = x^2 + y^2$ — (III)

After eliminating $x$ and $y$ from the eqn (I), (II) & (III), we get the required eqn.

Example: Find the pedal eqn of the parabola $y^2 = 4a(x+a)$.

Soln: Given that $y^2 = 4a(x+a)$ —— (i)

Differentiating w.r.t. $x$, we have

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y} \quad \text{—— (ii)}$$

Hence the tangent at $(x,y)$ is given by

$$Y - y = \left(\frac{2a}{y}\right)(X - x) \quad \text{—— (iii)}$$

So $$p = \frac{x(2a/y) - y}{\sqrt{1+\left(\frac{2a}{y}\right)^2}} = \frac{2ax/y - y}{\sqrt{(y^2+4a^2)/y^2}}$$

$$= \frac{2ax - y^2}{\sqrt{y^2+4a^2}} = \frac{2ax - 4a(x+a)}{\sqrt{4ax+4a^2+4a^2}}$$

$$p = -\sqrt{\{a(x+2a)\}}$$

$$\Rightarrow p^2 = a(x+2a)^{2\cdot 1/2} = a(x+2a) \quad \text{—— (iv)}$$

& $r^2 = x^2+y^2 = x^2+4a(x+a) = (x+2a)^2$ —— (v)

using $r = x+2a$ in (iv), we have

$\boxed{p^2 = ar}$ which is the required pedal equation.