B.Sc. Ist Year Maths Hons.
(B.Sc. IInd Year Maths Subs.)
Diff. Calculus (Paper IInd)

Tangents & Normals:

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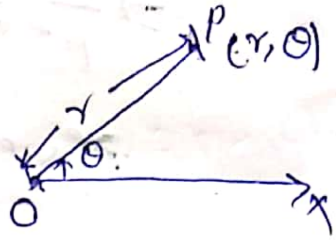
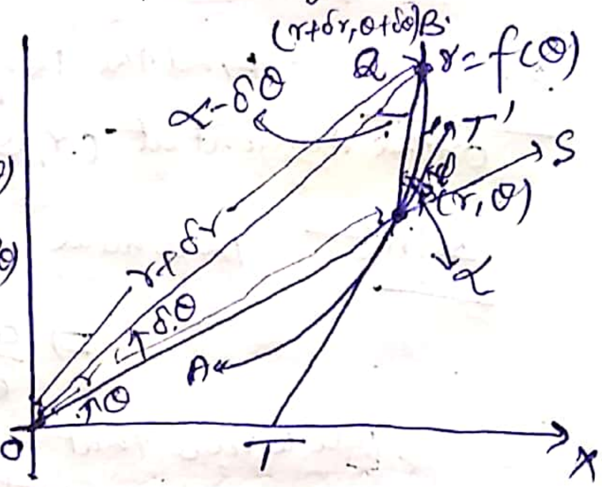
Angle between radius vector and tangent:

Let O be a fixed point called the pole. and OX, a fixed line. called the initial line. The position of a point 'P' is determined relative to O, if θ, the magnitude of angle XOP, and the r length of OP. Then (r, θ) are said to be polar co-ordinate of P, r and θ are called the radius vector and vectorial angle respectively.



Let $r = f(\theta)$ be the curve. and $P(r, \theta)$ be the any point of the curve $r = f(\theta)$. Let another point $Q(r+\delta r, \theta+\delta\theta)$ in neighbourhood of $P(r, \theta)$ either side on the curve. Produce OP to S and tangent TP to T' and



angle between radius vector OP and tangent PT is ϕ i.e. $\angle SPT' = \phi$. Let $\angle SPQ = \alpha \to \phi$; as $Q \to P$.

In $\triangle OPQ$; $\angle OPQ = \pi - \alpha$; $\angle PQO = \alpha - \delta\theta$.

By sine rule we have $\dfrac{OQ}{OP} = \dfrac{\sin\angle OPQ}{\sin\angle PQO}$

By figure and our supposition, we have.

$$\frac{r+\delta r}{r} = \frac{\sin(\pi - \alpha)}{\sin(\alpha - \delta\theta)} = \frac{\sin\alpha}{\sin(\alpha - \delta\theta)}$$

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$$\frac{r+\delta r}{r} = \frac{\sin\alpha}{\sin(\alpha-\delta\theta)}$$

$$\Rightarrow 1+\frac{\delta r}{r} = \frac{\sin\alpha}{\sin(\alpha-\delta\theta)}$$

$$\Rightarrow \frac{\delta r}{r} = \frac{\sin\alpha}{\sin(\alpha-\delta\theta)} - 1 = \frac{\sin\alpha-\sin(\alpha-\delta\theta)}{\sin(\alpha-\delta\theta)}$$

$$= \frac{2\cos\left(\frac{2\alpha-\delta\theta}{2}\right)\cdot\sin\left(\frac{\delta\theta}{2}\right)}{\sin(\alpha-\delta\theta)}$$

$$= \frac{2\cos\left(\alpha-\frac{\delta\theta}{2}\right)\cdot\sin\left(\frac{\delta\theta}{2}\right)}{\sin(\alpha-\delta\theta)}$$

$$\Rightarrow \frac{1}{r}\cdot\frac{\delta r}{\left(\frac{\delta\theta}{2}\right)} = 2\frac{\cos(\alpha-\delta\theta/2)}{\sin(\alpha-\delta\theta)}\cdot\frac{\sin\left(\frac{\delta\theta}{2}\right)}{\left(\frac{\delta\theta}{2}\right)}$$

$$\Rightarrow \frac{1}{r}\cdot\frac{\delta r}{\delta\theta} = \frac{\cos(\alpha-\delta\theta/2)}{\sin(\alpha-\delta\theta)}\cdot\left\{\frac{\sin\delta\theta/2}{\frac{\delta\theta}{2}}\right\}$$

As $Q\to P$, then $\delta\theta\to 0$; $\delta r\to 0$ and $\alpha\to\phi$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\cos\phi}{\sin\phi}\cdot 1$$

$$\Rightarrow \tan\phi = r\frac{d\theta}{dr}.$$

$$\Rightarrow \phi = \tan^{-1}\left\{r\cdot\frac{d\theta}{dr}\right\}.$$

which is required angle.

Example:– Find the angle at which the radius vector cuts the $\frac{l}{r} = 1+e\cos\theta$.

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Solution: Given curve is $\frac{l}{r} = 1 + e\cos\theta$

Taking log both sides, we have

$$\log\left(\frac{l}{r}\right) = \log(1 + e\cos\theta)$$

Now differentiating w.r.t. θ, we have.

$$\frac{1}{\left(\frac{l}{r}\right)} \cdot \left(-\frac{l}{r^2}\right) \cdot \frac{dr}{d\theta} = \frac{1}{(1+e\cos\theta)} \cdot e \cdot (-\sin\theta).$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta}.$$

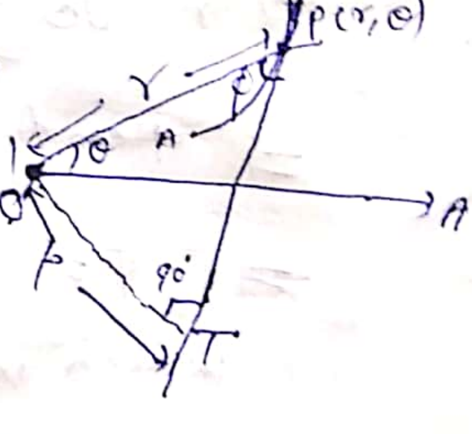
$$\Rightarrow r\frac{d\theta}{dr} = \frac{1+e\cos\theta}{e\sin\theta}$$

$$\Rightarrow \tan\phi = \frac{1+e\cos\theta}{e\sin\theta}$$

$$\Rightarrow \phi = \tan^{-1}\left\{\frac{1+e\cos\theta}{1+e\sin\theta}\right\}.$$

Length of the perpendicular from pole to the tangent:

Let $r = f(\theta)$ be the curve. If p be the length of the $\perp^{ar}$ OT from the pole O to the tangent at any point P on the curve; then from the side figure, in ΔOPT, we have



$$\frac{p}{r} = \sin\phi$$

$$\Rightarrow \frac{1}{p} = \frac{1}{r\sin\phi} \Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} = \frac{1}{r^2}\operatorname{cosec}^2\phi.$$

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$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$= \frac{1}{r^2}\left\{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{--- (1)}$$

If we let $u = \frac{1}{r}$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\cdot\frac{dr}{d\theta}$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

Using in (1); we have $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

Example:- Find the length of the $\perp^{r}$ from the pole on the tangent at (r,θ) of the curve $r = a(1-\cos\theta)$.

Soln:- We have $r = a(1-\cos\theta)$

$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$$

We know that $\tan\phi = r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta}$

$$\Rightarrow \tan\phi = \tan\theta/2$$

$$\Rightarrow \phi = \theta/2$$

Hence $p = r\sin\phi = a(1-\cos\theta).\sin\theta/2$

$$= 2a\sin^3\theta/2.$$

Exercise:- Find the angle ϕ for the curve

$$a\theta = \sqrt{(r^2-a^2)} - a\cos^{-1}\left(\frac{a}{r}\right).$$

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