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B.Sc (Part-1) Hons paper-1

General problem of Small oscillations in one degree of freedom

In order to Salve the Small oscillation problem, we Consider ~~that~~ the deflection of the System from the position of equilibrium which is the position of the system generalised force acting on the system vanishes.

$$\therefore \; Q_j = -\left(\frac{\partial v}{\partial q_j}\right) = 0 \quad \text{——— (1)}$$

which is Shows that the potential energy is extremum at the equilibrium Configuration of the System.

In one degree of freedom there is only one generalised Co-ordinate q and ~~and~~ above equation reduces to

$$\left(\frac{\partial v}{\partial q}\right)_0 = 0 \quad \text{——— (2)}$$

For one degree of freedom Taylor expansion for v is

$$v(q) = v(q_0) + \left(\frac{\partial v}{\partial q}\right)_0 (q-q_0) + \frac{1}{2}\left(\frac{\partial^2 v}{\partial q^2}\right)_0 (q-q_0)^2 + \text{-----}$$

for which the potential energy becomes

$$V(q) = \frac{\beta}{2}(q-q_0)^2 \quad \text{—— (3)}$$

where $\beta = \left(\frac{\partial^2 V}{\partial q^2}\right)_0$ —— (4)

Thus the force near equilibrium is

$$F(q) = -\frac{\partial V}{\partial q} = -\beta(q-q_0)$$

For this force to be restoring force, the following in equality must hold,

$$\beta = \left(\frac{\partial^2 V}{\partial q^2}\right)_0 > 0 \quad \text{—— (6)}$$

which is stability Condition for equilibrium. This Condition in addition (2) shows that the potential energy at equilibrium must be minimum $\therefore \theta = 0$

The kinitic energy of a particle in Cartesian Co-ordinate is

$$T = \frac{1}{2}m(x^2+y^2+z^2) \quad \text{—— (7)}$$

for one degree of freedom, let us put

$$x = x(q),\ y = y(q) \text{ and } z = z(q)$$

then

$$T = \frac{1}{2}m\left[\left(\frac{dx}{dq}\right)^2+\left(\frac{dy}{dq}\right)^2+\left(\frac{dz}{dq}\right)^2\right]q^2$$

$$= \frac{1}{2}\alpha(1)q^2 \quad \text{— (8)}$$

where $\alpha = \frac{m}{2}\left[\left(\frac{dx}{dq}\right)^2+\left(\frac{dy}{dq}\right)^2+\left(\frac{dz}{dq}\right)^2\right]$

On expanding $\alpha(q)$ in Taylor's Series about $q = q_0$ we get

$$T = \frac{1}{2}\alpha(q_0)\dot{q}^2 + \frac{1}{2}\left(\frac{\partial \alpha}{\partial q}\right)_{q=q_0}(q-q_0)^2 + \cdots \quad (9)$$

From which the equation Corresponding to equation

$$T = \frac{1}{2}\sum_{ij} m_{ij}(q_{01}, \cdots q_{0n})\dot{\eta}_j\dot{\eta}_j$$ is derived as

$$T = \frac{1}{2}\alpha(q_0)\dot{q}^2 = \frac{1}{2}\alpha(0)\dot{q}^2 \text{ (putting } q_0 = 0) \quad (10)$$

where the terms other than first in equation (9) have been neglected since the term $\alpha(q_0)\dot{q}^2$ is already of Some order of smallness for Small oscillations as the Second term in the expansion of $V(q)$.

From equation (3) and equation (10) the Lagrangian Can be written as

$$L = \frac{1}{2}\alpha(0)\dot{q}^2 - \frac{\beta}{2}q^2 \quad (11)$$

$$\therefore \frac{\partial L}{\partial q} = -\beta q \text{ and } \frac{\partial L}{\partial \dot{q}} = \alpha(0)\dot{q}$$

and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \alpha(0)\ddot{q}$

Lagrange's equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0$

or, $\alpha(0)\ddot{q} - (-\beta q) = 0$ or, $\alpha(0)\ddot{q} + \beta q = 0$

or, $\ddot{q} + \frac{\beta}{\alpha(0)} q = 0$

putting $\omega^2 = \beta/\alpha(0)$ we get

$$\ddot{q} + \omega^2 q \quad (12)$$

This equation represents the general form for the oscillation in one degree freedom. This is the equation of Simple Harmonic motion.