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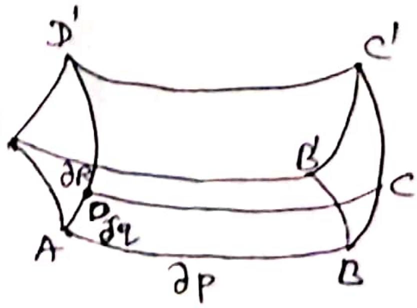
Laplace's equation in term of Spherical polar Co-ordinates.

Laplace's equation can easily be deduced in cylindrical and spherical polar co-ordinates from generalised orthogonal curvilinear co-ordinates. A system of orthogonal curvilinear co-ordinates is one which corresponds to the points of intersection of a triply orthogonal system of three families of surfaces.

Let the curvilinear co-ordinates be P, Q, R. These co-ordinates depend on the rectangular co-ordinates x, y, z, i.e, every one of P, Q, R has three rectangular components. So the surface in these co-ordinates consists of three families expression by

$$P(x,y,z) = \text{Constant}$$
$$q(x,y,z) = \text{Constant}$$
$$R(x,y,z) = \text{Constant}$$



These three families are mutually perpendicular. The volume element bounded by these surfaces. Let E_P, E_q and E_R be the electric intensities in the direction of increasing p, q, R respectively and ∂p, ∂q and ∂R the increments in p, q and R respectively. in order to get distances corresponding to these Co-ordinates.

$\therefore$ Area of ABCD $= a\partial p \cdot b\partial q$

$= ab\partial p\partial q$

$\therefore$ The flux passing through it $= E_R ab\partial p\partial q$

and the flux through A'B'C'D' outwards is given by

$$\left[E_R + \frac{\partial}{\partial R}(E_R)\partial R\right]\partial p \cdot \partial q \cdot ab$$

$\therefore$ Net outward flux $= \frac{\partial}{\partial R}(E_R)\partial R \cdot \partial p \cdot \partial q \cdot ab$ —— (1)

Similarly, the net outward flux through other two pair of surfaces are

$$\frac{\partial}{\partial p}(E_p)\partial p\,\partial q\,\partial R \cdot cb \quad \text{—— (2)}$$

and $$\frac{\partial}{\partial q}(E_q)\partial p\,\partial q\,\partial R \cdot ac \quad \text{—— (3)}$$

Adding equation (1), (2) and (3). we obtain the net flux diverging through the volume, which is Zero but some mutual charges are present in the volume.

$$\therefore \left[\frac{\partial}{\partial p}(E_p)bc + \frac{\partial}{\partial q}(E_q)ac + \frac{\partial}{\partial R}(E_R)ab\right] = 0$$

But
$$E_p = -\frac{1}{a} \cdot \frac{\partial v}{\partial p}$$

$$E_q = -\frac{1}{b} \cdot \frac{\partial v}{\partial q}$$

and $E_R = -\frac{1}{c}\cdot\frac{\partial v}{\partial R}$

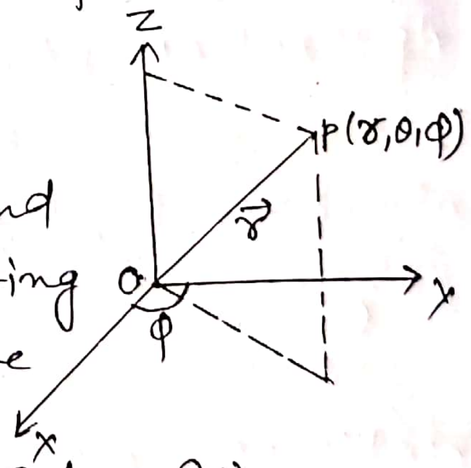
Hence the above equation becomes

$$\frac{\partial}{\partial p}\left(-\frac{1}{a}\right)\frac{\partial v}{\partial p}bc + \frac{\partial}{\partial q}\left(-\frac{1}{b}\cdot\frac{\partial v}{\partial q}\right)ac + \frac{\partial}{\partial R}\left(-\frac{1}{c}\frac{dv}{\partial R}\right)ab = 0$$

or, $$\frac{\partial}{\partial p}\left[\frac{bc}{a}\cdot\frac{\partial v}{\partial p}\right] + \frac{\partial}{\partial q}\left[\frac{ac}{b}\cdot\frac{\partial v}{\partial q}\right] + \frac{\partial}{\partial R}\left[\frac{ab}{c}\frac{\partial v}{\partial R}\right] = 0 \quad (4)$$

This ~~is~~ equation is the most general expression for Laplace's equation. This expression can be expressed in spherical expression and cylindrical co-ordinates.

Laplace's equation in spherical co-ordinates



in case p, q, R are represented by r, θ, φ and $a = 1, b = r, c = r\sin\theta$. putting these values in eqn (4) we obtain

$$\nabla^2 V = \frac{\partial}{\partial r}\left(r^2\sin\theta\,\frac{\partial v}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial v}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\cdot\frac{\partial v}{\partial\phi}\right) = 0$$

or, $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\sin\theta\, d\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 v}{\partial\phi^2} = 0$$